Application of New Triangular Functions to Nonlinear Partial Differential Equations

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The results of some new research on a new class of triangular functions that unite the characteristics of the classical triangular functions are presented. Taking into consideration the great role played by triangular functions in geometry and physics, it is possible to expect that the new theory of the triangular functions will bring new results and interpretations in mathematics, biology, physics and cosmology. New traveling wave solutions of some nonlinear partial differential equations are obtained in a unified way. The main idea of this method is to express the solutions of these equations as a polynomial in the solution of the Riccati equation that satisfy the symmetrical triangular Fibonacci functions. We apply this method to the combined Korteweg-de Vries (KdV) and modified KdV (mKdV) equations, the generalized Kawahara equation, Ito's 5th-order mKdV equation and Ito's 7th-order mKdV equation.

Key words: Exact Solutions; Triangular Fibonacci Functions; Nonlinear Evolution Equations; Traveling Wave Solutions.
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1. Introduction

It is well known that nonlinear partial differential equations (NLPDEs) are widely used to describe complex phenomena in various fields of sciences, particularly in physics. The exact traveling wave solution of NLPDEs is one of the fundamental objects of study in mathematical physics. To find mathematical models for the phenomena, the investigation of exact solutions of NLPDEs will help to have a better understanding of these physical phenomena. In recent years, various powerful methods have been developed to construct exact solitary wave solutions and periodic wave solutions of the nonlinear evolution equations (NLEEs), such as: the tanh function method [1, 2], the extended tanh function method [3], the Jacobi elliptic function expansion method [4], the F-expansion method [5], the generalized Jacobi elliptic function method [6] and other methods [7 – 11]. The symbolic software programs have been presented [12, 13] to find exact solutions of NLPDEs in terms of hyperbolic and elliptic functions.

In [14], Conte and Musette presented an indirect method to seek some solitary wave solutions of NLPDEs that can be expressed as a polynomial in two elementary functions which satisfy a projective Riccati system [15]. By use of this method, some solitary wave solutions of many NLPDEs have been obtained [14, 16]. Recently, Yan [17] and Chen and Li [18] further developed the Conte and Musette method by introducing a more general projective Riccati equation and obtained many exact traveling wave solutions of some NLPDEs.

The finding of a new mathematical algorithm to construct exact solutions of NLPDEs is important and might have significant impact on future research. In [19], we constructed symmetrical hyperbolic Fibonacci functions and found new solutions of the Riccati equation by using these functions. Also, we devised an algorithm called Fibonacci Riccati method to obtain new exact solutions of NLPDEs. Here, we introduce new triangular functions. We call them symmetrical triangular Fibonacci functions and use them to obtain new solutions of the Riccati equation.

The present paper is organized as follows. In the next section, we introduce the symmetrical triangular Fibonacci functions and their properties. In Section 3, we introduce the triangular Fibonacci Riccati (TFR) method to NLPDEs. In Section 4, we apply the TFR method to NLPDEs such as the combined Korteweg-de Vries (KdV) and modified KdV (mKdV) equation, the generalized Kawahara equation, Ito's 5th-order mKdV equation and Ito's 7th-order mKdV equation. Finally, we give some features and comments.

2. Definition and Properties of the Symmetrical Triangular Fibonacci Functions

We know that the symmetrical hyperbolic Fibonacci sine (sFs) function, the symmetrical hyperbolic Fibonacci cosine (cFs) function and the symmetrical hyperbolic Fibonacci tangent (tFs) function are defined [20] as

$$\mathrm{sFs}(x)=\frac{\alpha^{x}-\alpha^{-x}}{\sqrt{5}},\quad \mathrm{cFs}(x)=\frac{\alpha^{x}+\alpha^{-x}}{\sqrt{5}},\quad \mathrm{tFs}(x)=\frac{\alpha^{x}-\alpha^{-x}}{\alpha^{x}+\alpha^{-x}}. \tag{1}$$

They are introduced to consider so-called symmetrical representations of the hyperbolic Fibonacci functions and they may present a certain interest for modern theoretical physics taking into consideration the great role played by the Golden Section, Golden Proportion, Golden Ratio, Golden Mean in modern physical research [20]. The symmetrical Fibonacci hyperbolic cotangent (cot Fs) function is $\cot\mathrm{Fs}(x)=\frac{1}{\mathrm{tFs}(x)}$, the symmetrical hyperbolic Fibonacci secant (sec Fs) function is $\sec\mathrm{Fs}(x)=\frac{1}{\mathrm{cFs}(x)}$, and the symmetrical hyperbolic Fibonacci cosecant (csc Fs) function is $\csc\mathrm{Fs}(x)=\frac{1}{\mathrm{sFs}(x)}$. These functions satisfy the following relations [20]:

$$\mathrm{cFs}^2(x)-\mathrm{sFs}^2(x)=\frac{4}{5},\quad 1-\mathrm{tFs}^2(x)=\frac{4}{5}\sec\mathrm{Fs}^2(x),\quad \cot\mathrm{Fs}^2(x)-1=\frac{4}{5}\csc\mathrm{Fs}^2(x). \tag{2}$$

Also, from the above definitions, we give the derivative formulas of the symmetrical hyperbolic Fibonacci functions as follows:

$$\frac{\mathrm{d}\,\mathrm{sFs}(x)}{\mathrm{d}x}=\mathrm{cFs}(x)\ln\alpha,\quad \frac{\mathrm{d}\,\mathrm{cFs}(x)}{\mathrm{d}x}=\mathrm{sFs}(x)\ln\alpha,\quad \frac{\mathrm{d}\,\mathrm{tFs}(x)}{\mathrm{d}x}=\frac{4}{5}\sec\mathrm{Fs}^2(x)\ln\alpha. \tag{3}$$

The above symmetrical hyperbolic Fibonacci functions are connected with the classical hyperbolic functions by the simple correlations

$$\mathrm{sFs}(x)=\frac{2}{\sqrt{5}}\sinh(x\ln\alpha),\quad \mathrm{cFs}(x)=\frac{2}{\sqrt{5}}\cosh(x\ln\alpha),\quad \mathrm{tFs}(x)=\tanh(x\ln\alpha). \tag{4}$$

From the above definitions and properties of the symmetrical hyperbolic Fibonacci functions we can define the symmetrical triangular Fibonacci sine (sTFs) function, the symmetrical triangular Fibonacci cosine (cTFs) function, and the symmetrical triangular Fibonacci tangent (tTFs) function as

$$\mathrm{sTFs}(x)=\frac{\alpha^{\mathrm{i}x}-\alpha^{-\mathrm{i}x}}{\mathrm{i}\sqrt{5}},\quad \mathrm{cTFs}(x)=\frac{\alpha^{\mathrm{i}x}+\alpha^{-\mathrm{i}x}}{\mathrm{i}\sqrt{5}},\quad \mathrm{tTFs}(x)=\frac{\mathrm{sTFs}(x)}{\mathrm{cTFs}(x)}. \tag{5}$$

The symmetrical triangular Fibonacci cotangent (cot TFs) function is $\cot\mathrm{TFs}(x)=\frac{1}{\mathrm{tTFs}(x)}$, the symmetrical triangular Fibonacci secant (sec TFs) function is $\sec\mathrm{TFs}(x)=\frac{1}{\mathrm{cTFs}(x)}$, and the symmetrical triangular Fibonacci cosecant (csc TFs) function is $\csc\mathrm{TFs}(x)=\frac{1}{\mathrm{sTFs}(x)}$. These functions satisfy the following relations [20]:

$$\mathrm{cTFs}^2(x)+\mathrm{sTFs}^2(x)=\frac{4}{5},\quad 1+\mathrm{tTFs}^2(x)=\frac{4}{5}\sec\mathrm{TFs}^2(x),\quad \cot\mathrm{TFs}^2(x)+1=\frac{4}{5}\csc\mathrm{TFs}^2(x). \tag{6}$$

Also, from the above definitions, we give the derivative formulas of the symmetrical triangular Fibonacci functions as follows:

$$\frac{\mathrm{d}\,\mathrm{sTFs}(x)}{\mathrm{d}x}=\mathrm{cTFs}(x)\ln\alpha,\quad \frac{\mathrm{d}\,\mathrm{cTFs}(x)}{\mathrm{d}x}=-\mathrm{sTFs}(x)\ln\alpha,\quad \frac{\mathrm{d}\,\mathrm{tTFs}(x)}{\mathrm{d}x}=\frac{4}{5}\sec\mathrm{TFs}^2(x)\ln\alpha. \tag{7}$$

The above symmetrical triangular Fibonacci functions are connected with the classical triangular functions by the simple correlations

$$\mathrm{sTFs}(x)=\frac{2}{\sqrt{5}}\sin(x\ln\alpha),\quad \mathrm{cTFs}(x)=\frac{2}{\sqrt{5}}\cos(x\ln\alpha),\quad \mathrm{tTFs}(x)=\tan(x\ln\alpha). \tag{8}$$

3. The Triangular Fibonacci Riccati Method

The main idea of this method is to express the solution of an NLPDE as a polynomial in the solution of the Riccati equation that satisfies the symmetrical triangular Fibonacci functions. Consider a given NLPDE

$$H(u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots)=0. \quad (9)$$

The TFR method for solving (9) proceeds in the following four steps:

Step 1. We seek the traveling wave solution of (9) in the form

$$u(x,t)=u(\xi), \quad \xi=k(x-\omega t), \quad (10)$$

where k and ω are the wave number and wave velocity, respectively. Substituting (10) into (9) yields the ordinary differential equation (ODE)

$$\tilde{H}(u,u',u'',u''',\ldots)=0, \quad u'=\frac{\mathrm{d}u}{\mathrm{d}\xi},\ldots \text{ etc.}, \quad (11)$$

where $\tilde{H}$ is a polynomial of u and its various derivatives. If $\tilde{H}$ is not a polynomial of u and its various derivatives, then we may use new variables $v=v(\xi)$ which make $\tilde{H}$ to become a polynomial of v and its various derivatives.

Step 2. Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$:

$$u(\xi)=\sum_{i=0}^{n}a_iF^i(\xi), \quad a_n\neq 0, \quad (12)$$

where n is the highest degree of the series, which can be determined by balancing the highest derivative term (or terms) with the nonlinear term (or terms) in (11), and a_i are some parameters to be determined. The function $F(\xi)$ satisfies the Riccati equation

$$F'(\xi)=A+BF^2(\xi), \quad '\equiv\frac{\mathrm{d}}{\mathrm{d}\xi}, \quad (13)$$

where A and B are constants.

Step 3. Substituting (12) with (13) into the ODE (11), the left-hand side of (11) can be converted into a polynomial in $F(\xi)$. Setting each coefficient of the polynomial to zero yields a system of algebraic equations for $a_0,a_1,a_2,\ldots,a_n,k$ and ω.

Step 4. Solving the system obtained in step 3, $a_0,a_1,a_2,\ldots,a_n,k$ and ω can be expressed by A and B. Substituting these results into (12), a general formula of traveling wave solutions of (9) can be obtained. A and B in ODE (13) have to be choosen properly such that the corresponding solution $F(\xi)$ of it is one of the symmetrical triangular Fibonacci functions given bellow.

Case 1. If $A=\ln\alpha$ and $B=\ln\alpha$, then (13) possesses the solution $\mathrm{tTFs}(\xi)$.

Case 2. If $A=\ln\alpha$ and $B=-\ln\alpha$, then (13) possesses the solution $\cot\mathrm{TFs}(\xi)$.

Case 3. If $A=\frac{\ln\alpha}{2}$ and $B=\frac{\ln\alpha}{2}$, then (13) possesses the solutions $\mathrm{tTFs}(\xi)\pm\sec\mathrm{TFs}(\xi)$, $\frac{\mathrm{tTFs}(\xi)}{1\pm\sec\mathrm{TFs}(\xi)}$, $\csc\mathrm{TFs}(\xi)-\cot\mathrm{TFs}(\xi)$.

Case 4. If $A=-\frac{\ln\alpha}{2}$ and $B=-\frac{\ln\alpha}{2}$, then (13) possesses the solutions $\cot\mathrm{TFs}(\xi)\pm\csc\mathrm{TFs}(\xi)$, $\frac{\cot\mathrm{TFs}(\xi)}{1\pm\csc\mathrm{TFs}(\xi)}$, $\sec\mathrm{TFs}(\xi)-\mathrm{tTFs}(\xi)$.

Case 5. If $A=\ln\alpha$ and $B=4\ln\alpha$, then (13) possesses the solution $\frac{\mathrm{tTFs}(\xi)}{1-\mathrm{tTFs}^2(\xi)}$.

Case 6. If $A=-\ln\alpha$ and $B=-4\ln\alpha$, then (13) possesses the solution $\frac{\cot\mathrm{TFs}(\xi)}{1-\cot\mathrm{TFs}^2(\xi)}$.

Now, we can apply the TFR method to some NLPDEs.

4. Applications

4.1. The Combined KdV and mKdV Equation

We consider the combined KdV and mKdV equation

$$u_t+6auu_x+6bu^2u_x+cu_{xxx}=0 \quad (14)$$

with the constants A, b and c. Equation (14) is widely used in various fields such as solid-state physics, plasma physics, fluid physics and quantum field theory [21, 22]. It is clear that (14) is a combination of the KdV and mKdV equations. As a result the combined KdV and mKdV equation is also integrable, which means that it has a Bäcklund transformation, a bilinear form, a Lax pair and an infinite number of conservation laws etc. The periodic wave solutions of (14) have been studied in [23].

Now, we can apply the TFR method to the combined KdV and mKdV equation (14). Substituting (10) into (14) yields

$$-\omega u'+6auu'+6bu^2u'+ck^2u'''=0. \quad (15)$$

Balancing u''' with u^2u' gives $n=1$. Therefore, the solution of (15) can be expressed as

$$u = a_0 + a_1 F(\xi). \tag{16}$$

With the help of the symbolic software Maple, substituting (16) into (15) yields a set of algebraic equations with respect to $F^i(\xi)$. We set the coefficients of $F^i(\xi)$ $(i=0,1,2,3,4)$ in the obtained equation to zero. We further obtain a system of algebraic equations. Solving this set of equations for a_0, a_1, k and ω with the aid of Maple, we find

$$a_0 = -\frac{a}{2b}, \quad a_1 = \pm kB\sqrt{\frac{-c}{b}}, \quad \omega = 2ck^2BA - \frac{3a^2}{2b}, \tag{17}$$

where k is an arbitrary constant. Thus, we obtain the general formulae of the solutions of the combined KdV and mKdV equation (14):

$$u = -\frac{a}{2b} \pm kB\sqrt{\frac{-c}{b}}F(\xi), \quad \xi = k\left(x - \left(2ck^2BA - \frac{3a^2}{2b}\right)t\right), \quad bc<0. \tag{18}$$

Selecting some special values of A, B and the corresponding function $F(\xi)$, we have the following traveling wave solutions of (14):

$$u_1 = -\frac{a}{2b} \pm k\ln\alpha\sqrt{\frac{-c}{b}}\,\mathrm{tTFs}(\xi), \quad \xi = k\left(x - \left(2ck^2\ln\alpha^2 - \frac{3a^2}{2b}\right)t\right), \quad bc<0, \tag{19}$$

$$u_2 = -\frac{a}{2b} \mp k\ln\alpha\sqrt{\frac{-c}{b}}\cot\mathrm{TFs}(\xi), \quad \xi = k\left(x + \left(2ck^2\ln\alpha^2 - \frac{3a^2}{2b}\right)t\right), \quad bc<0, \tag{20}$$

$$u_3 = -\frac{a}{2b} \pm \frac{k\ln\alpha}{2}\sqrt{\frac{-c}{b}}[\mathrm{tTFs}(\xi) \pm \sec\mathrm{TFs}(\xi)],$$

$$u_4 = -\frac{a}{2b} \pm \frac{k\ln\alpha}{2}\sqrt{\frac{-c}{b}}[\csc\mathrm{TFs}(\xi) - \cot\mathrm{TFs}(\xi)],$$

$$u_5 = -\frac{a}{2b} \pm \frac{k\ln\alpha}{2}\sqrt{\frac{-c}{b}}\left[\frac{\mathrm{tTFs}(\xi)}{1\pm\sec\mathrm{TFs}(\xi)}\right],$$

$$u_6 = -\frac{a}{2b} \mp \frac{k\ln\alpha}{2}\sqrt{\frac{-c}{b}}[\cot\mathrm{TFs}(\xi) \pm \csc\mathrm{TFs}(\xi)],$$

$$u_7 = -\frac{a}{2b} \mp \frac{k\ln\alpha}{2}\sqrt{\frac{-c}{b}}[\sec\mathrm{TFs}(\xi) - \mathrm{tTFs}(\xi)],$$

$$u_8 = -\frac{a}{2b} \mp \frac{k\ln\alpha}{2}\sqrt{\frac{-c}{b}}\left[\frac{\cot\mathrm{TFs}(\xi)}{1\pm\csc\mathrm{TFs}(\xi)}\right],$$

$$\xi = k\left(x - \left(\frac{ck^2}{2}\ln\alpha^2 - \frac{3a^2}{2b}\right)t\right), \quad bc<0, \tag{21}$$

$$u_9 = -\frac{a}{2b} \pm 4k\ln\alpha\sqrt{\frac{-c}{b}}\left[\frac{\mathrm{tTFs}(\xi)}{1-\mathrm{tTFs}^2(\xi)}\right],$$

$$u_{10} = -\frac{a}{2b} \pm 4k\ln\alpha\sqrt{\frac{-c}{b}}\left[\frac{\cot\mathrm{TFs}(\xi)}{1-\cot\mathrm{TFs}^2(\xi)}\right],$$

$$\xi = k\left(x - \left(8ck^2\ln\alpha^2 - \frac{3a^2}{2b}\right)t\right), \quad bc<0. \tag{22}$$

Figures 1a–d show the characters of the new solutions u_1, u_3, u_5, and u_7, respectively, with $a=3$, $b=1$, $c=-2$, and $k=0.25$. It is easily seen that the obtained solutions are periodic ones.

4.2. The Generalized Kawahara Equation

We consider the generalized Kawahara equation

$$u_t + \sigma u u_x + u^2 u_{xx} + u_{xxx} - u_{xxxxx} = 0, \tag{23}$$

where σ is a real constant. The generalized Kawahara equation describes many different physical phenomena, for example in the theory of magneto-acoustic waves in plasmas [24].

Now, we can apply the TFR method to the generalized Kawahara equation (23). Substituting (10) into (23) yields

$$-\omega u' + \sigma u u' + k u^2 u'' + k^2 u'' - k^4 u'''' = 0. \tag{24}$$

Therefore, the solution of (23) can be expressed as

$$u = a_0 + a_1 F(\xi) + a_2 F^2(\xi). \tag{25}$$

With the help of the symbolic software Maple, substituting (25) into (24) yields a set of algebraic equations with respect to $F^i(\xi)$. We set the coefficients of $F^i(\xi)$ $(i=0,1,\ldots,7)$ in the obtained equation to zero. We further obtain a system of algebraic equations. Solving this set of equations for a_0, a_1, a_2, k and ω with the aid of Maple, we obtain the general formulae of the solution of the generalized Kawahara equation (23):

$$u = -\frac{\left(2-80k^2BA+\sigma\sqrt{10}\right)\sqrt{10}}{20} + 6\sqrt{10}B^2k^2F^2(\xi), \tag{26}$$

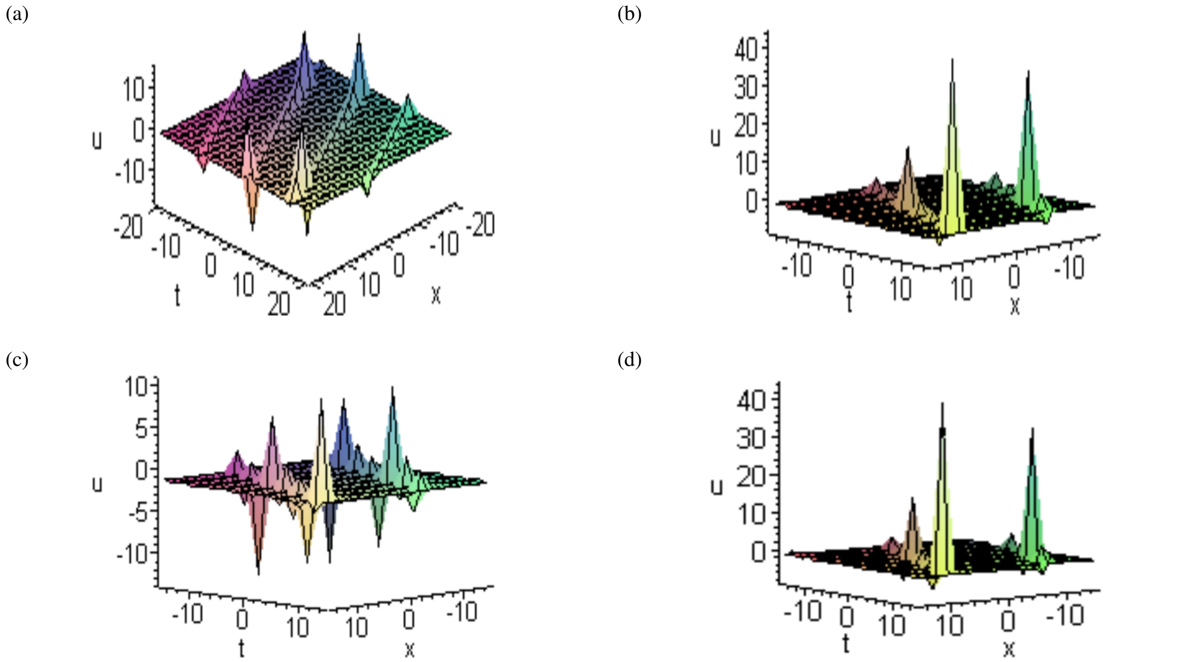

Fig. 1. The periodic solution of the combined KdV and mKdV equation (14) with $a=3$, $b=1$, $c=-2$, and $k=0.25$; (a) plots of u_1; (b) plots of u_3; (c) plots of u_5; (d) plots of u_7.

where $\xi = k\left(x-\left(\frac{1}{10}+24k^4B^2A^2-\frac{\sigma^2}{4}\right)t\right)$. By selecting the special values of A, B and the corresponding function $F(\xi)$, we have the following traveling wave solutions of the generalized Kawahara equation (23):

$$\begin{aligned} u_1 &= -\frac{\left(2-80k^2\ln\alpha^2+\sigma\sqrt{10}\right)\sqrt{10}}{20} \\ &\quad +6\sqrt{10}\ln\alpha^2k^2\mathrm{tTFs}^2(\xi), \\ u_2 &= -\frac{\left(2+80k^2\ln\alpha^2+\sigma\sqrt{10}\right)\sqrt{10}}{20} \\ &\quad +6\sqrt{10}\ln\alpha^2k^2\cot\mathrm{TFs}^2(\xi), \end{aligned} \tag{27}$$

with $\xi = k\left(x-\left(\frac{1}{10}+24k^4\ln\alpha^4-\frac{\sigma^2}{4}\right)t\right)$. The reminder solutions are omitted for simplicity. Figure 2 shows the characters of the new solutions of the generalized Kawahara equation (23) with $\sigma=1$ and $k=2.5$.

4.3. Ito's 5th-Order mKdV Equation

We consider Ito's 5th-order mKdV equation [24]

$$u_t+\left(6u^5+10\sigma(u^2u_{xx}+uu_x^2)+u_{xxxx}\right)_x=0, \tag{28}$$

where σ is a real constant. Now, we can apply the TFR

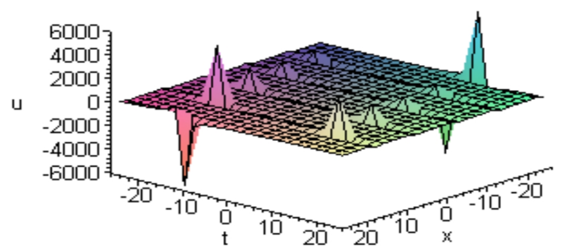

Fig. 2. The periodic solution of the generalized Kawahara equation (23) with $\sigma=1$ and $k=2.5$.

method to Ito's 5th-order mKdV equation (28). Substituting (10) into (28) yields

$$\begin{aligned} &-\omega u'+30u^4u'+10\sigma k^2(4uu'u''+u^2u'''+u'^3) \\ &+k^4u'''''=0. \end{aligned} \tag{29}$$

Therefore, the solution of Ito's 5th-order mKdV equation (28) can be expressed as

$$u=a_0+a_1F(\xi). \tag{30}$$

With the help of the symbolic software Maple, substituting (30) into (29) yields a set of algebraic equations with respect to $F^i(\xi)$. We set the coefficients of $F^i(\xi)$ $(i=0,1,\ldots,6)$ in the obtained equation to zero. We

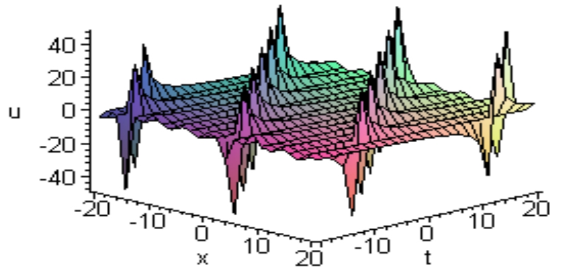

Fig. 3. The periodic solution of the 5th-order mKdV equation (28) with $\sigma = 1$ and $k = 2.5$.

further obtain a system of algebraic equations. Solving this set of equations for a_0, a_1, k and ω with the aid of Maple, we obtain:

$$\begin{aligned}&\text{Case 1.} \quad \sigma = -1, \quad a_0 = 0,\\ &\qquad \omega = 6k^4B^2A^2, \quad a_1 = \pm kB.\\ &\text{Case 2.} \quad \sigma = 1, \quad a_0 = 0,\\ &\qquad \omega = 6k^4B^2A^2, \quad a_1 = \mathrm{i}kB.\end{aligned} \tag{31}$$

Here k is an arbitrary constant and $\mathrm{i} = \sqrt{-1}$. Therefore, we obtain the general formulae of the solutions of Ito's 5th-order mKdV equation (28):

$$u = \pm kBF(k(x - 6k^4B^2A^2t)), \tag{32}$$

$$u = \mathrm{i}kBF(k(x - 6k^4B^2A^2t)). \tag{33}$$

With $\sigma = 1$, by selecting the special values of A, B and the corresponding function $F(\xi)$, we have the following traveling wave solutions of Ito's 5th-order mKdV equation (28):

$$\begin{aligned}u_1 &= \pm k\ln\alpha\,\mathrm{tTFs}(k(x - 6k^4\ln\alpha^4 t)),\\ u_2 &= \mp k\ln\alpha\cot\mathrm{TFs}(k(x - 6k^4\ln\alpha^4 t)),\end{aligned} \tag{34}$$

and with $\sigma = -1$, we have

$$\begin{aligned}u_3 &= \mathrm{i}k\ln\alpha\,\mathrm{tTFs}(k(x - 6k^4\ln\alpha^4 t)),\\ u_4 &= -\mathrm{i}k\ln\alpha\cot\mathrm{TFs}(k(x - 6k^4\ln\alpha^4 t)).\end{aligned} \tag{35}$$

The reminder solutions are omitted for simplicity. Figure 3 shows the characters of the new solutions of Ito's 5th-order mKdV equation (28) with $\sigma = 1$ and $k = 2.5$.

4.4. Ito's 7th-Order mKdV Equation

We consider Ito's 7th-order mKdV equation [24]

$$\begin{aligned}&u_t + \big(20\sigma u^7 + 17\sigma(u^4u_{xx} + 2u^3u_x^2)\\ &+14\sigma(u^2u_{xxxx} + 3uu_{xx}^2 + 4uu_xu_{xxx} + 5u_x^2u_{xx})\\ &+u_{xxxxxx}\big)_x = 0,\end{aligned} \tag{36}$$

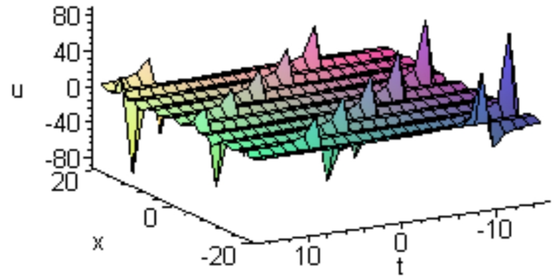

Fig. 4. The periodic solution of the 7th-order mKdV equation (36) with $\sigma = 1$ and $k = 2.5$.

where σ is a real constant. Now, we can apply the TFR method to Ito's 7th-order mKdV equation (36). Substituting (10) into (36) yields

$$\begin{aligned}&-\omega u' + 140\sigma u^6u'\\ &+70k^2(8u^3u'u'' + u^4u''' + 6u^2u'^3)\\ &+14\sigma^4k^4(6uu'u'''' + u^2u'''' + 13u'u''^2\\ &+10uu''u''' + 9u'^2u''') + k^6u'''''' = 0.\end{aligned} \tag{37}$$

Therefore, the solution of Ito's 7th-order mKdV equation (36) can be expressed as

$$u = a_0 + a_1F(\xi). \tag{38}$$

With the help of the symbolic software Maple, substituting (38) into (37) yields a set of algebraic equations with respect to $F^i(\xi)$. We set the coefficients of $F^i(\xi)$ $(i = 0, 1, \ldots, 8)$ in the obtained equation to zero. We further obtain a system of algebraic equations. Solving this set of equations for a_0, a_1, k and ω with the aid of Maple, we obtain:

$$\begin{aligned}&\text{Case 1.} \quad \sigma = -1, \quad a_0 = 0,\\ &\qquad \omega = 20k^6B^3A^3, \quad a_1 = \pm kB.\\ &\text{Case 2.} \quad \sigma = 1, \quad a_0 = 0,\\ &\qquad \omega = 20k^6B^3A^3, \quad a_1 = \mathrm{i}kB.\end{aligned} \tag{39}$$

Here k is an arbitrary constant and $\mathrm{i} = \sqrt{-1}$. Therefore, we obtain the general formulae of the solutions of Ito's 7th-order mKdV equation (36):

$$u = \pm kBF(k(x - 20k^6B^3A^3t)), \tag{40}$$

$$u = \mathrm{i}kBF(k(x - 20k^6B^3A^3t)). \tag{41}$$

With $\sigma = 1$, by selecting the special values of A, B and the corresponding function $F(\xi)$, we have the following traveling wave solutions of Ito's 7th-order mKdV equation (28):

$$u_1 = \pm k \ln\alpha \,\mathrm{tTFs}(k(x - 20k^6 \ln\alpha^6 t)),$$
$$u_2 = \mp k \ln\alpha \cot\mathrm{TFs}(k(x - 20k^6 \ln\alpha^6 t)), \quad (42)$$

and with $\sigma = -1$, we have

$$u_3 = \mathrm{i}k \ln\alpha \,\mathrm{tTFs}(k(x - 20k^6 \ln\alpha^6 t)),$$
$$u_4 = -\mathrm{i}k \ln\alpha \cot\mathrm{TFs}(k(x + 20k^6 \ln\alpha^6 t)). \quad (43)$$

The reminder solutions are omitted for simplicity. Figure 4 shows the characters of the new solutions of Ito's 7th-order mKdV equation (36) with $\sigma = 1$ and $k = 2.5$.

Remark 1. If $\alpha = \mathrm{e}$, the obtained solutions recover the solutions obtained by the tan function method, generalized hyperbolic function method and so on.

Remark 2. To the best of our knowledge, the solution using symmetrical triangular Fibonacci functions has not been found before.

Remark 3. To the best of our knowledge, the definitions of the symmetrical triangular Fibonacci functions have not been found before.

5. Summary and Discussion

We have proposed a TFR method and used it to construct new exact solutions of NLPDEs. The obtained solutions may be of important significance for the explanation of some practical physical problems. In contrast to the TFR method, there are some additional merits of our method. First, all the NLPDEs can be solved with our method more easily than with other tanh-function methods. More important, for some equations, with no extra effort we also picked up other new and more general types of solutions at the same time. Second, it is quite interesting that we choose A and B in a Riccati equation to show the number and types of traveling wave solutions for a NLPDE. Third, this method is also a computerized method, which allows to perform complicated and tedious algebraic calculation using a computer. The TFR method can be applied to other NLPDEs.

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